The Transmission Line Wave Equation

Let's assume that $v(z,t)$ and $i(z,t)$ each have the **time-harmonic** form:

$$v(z,t) = \text{Re}\left\{V(z)e^{j\omega t}\right\} \quad \text{and} \quad i(z,t) = \text{Re}\left\{I(z)e^{j\omega t}\right\}$$

The time-**derivative** of these functions are:

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\left\{j\omega V(z)e^{j\omega t}\right\}$$

$$\frac{\partial i(z,t)}{\partial t} = \text{Re}\left\{I(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\left\{j\omega I(z)e^{j\omega t}\right\}$$

Inserting these results into the telegrapher's equations, we find:

$$\text{Re}\left\{\frac{\partial V(z)}{\partial z} e^{j\omega t}\right\} = \text{Re}\left\{-(R + j\omega L) I(z)\, e^{j\omega t}\right\}$$

$$\text{Re}\left\{\frac{\partial I(z)}{\partial z} e^{j\omega t}\right\} = \text{Re}\left\{-(G + j\omega C) V(z)\, e^{j\omega t}\right\}$$

Simplifying, we have the **complex** form of **telegrapher's equations**:

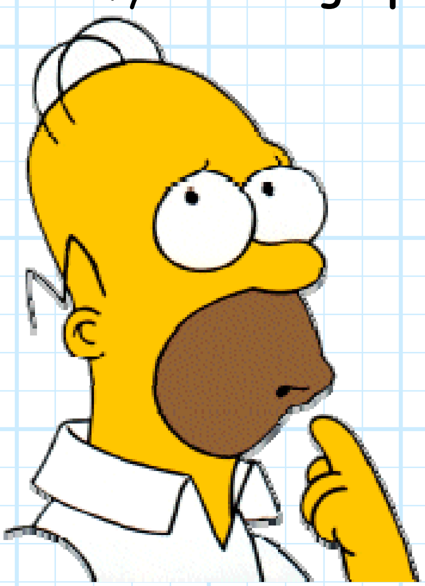
$$\boxed{\begin{aligned}\frac{\partial V(z)}{\partial z} &= -(R + j\omega L)\, I(z) \\[2ex] \frac{\partial I(z)}{\partial z} &= -(G + j\omega C)\, V(z)\end{aligned}}$$

Note that these complex differential equations are **not** a function of **time** t !

* The functions $I(z)$ and $V(z)$ are **complex,** where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position z.

* **Remember**, not just **any** function $I(z)$ and $V(z)$ can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.



Our task, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!

Q: *So, what functions $I(z)$ and $V(z)$* ***do*** *satisfy both telegrapher's equations??*

A: To make this easier, we will combine the telegrapher equations to form **one** differential equation for $V(z)$ and **another** for $I(z)$.

First, take the **derivative** with respect to z of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z}=-(R+j\omega L)\,I(z)\right\}$$
$$=\frac{\partial^2 V(z)}{\partial z^2}=-(R+j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z}=-(G+j\omega C)\,V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2}=(R+j\omega L)(G+j\omega C)\,V(z)$$

We can simplify this equation by defining the complex value γ:

$$\boxed{\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}}$$

So that:

$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 \, V(z)$$

In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 \, I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\boxed{\begin{gathered}\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 \, V(z) \\[2em] \frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 \, I(z)\end{gathered}}$$

These are known as the **transmission line wave equations**.

Note that value γ is **complex**, and is determined by taking the **square-root** of a **complex** value. Likewise, γ^2 is a **complex** value. Do **you** know how to square a complex number? Can **you** determine the square root of a complex number?

Note only **special** functions satisfy these wave equations; if we take the double derivative of the function, the result is the **original function** (to within a constant γ^2)!

Q: *Yeah right! Every function that **I** know is **changed** after a double differentiation. What kind of "magical" function could possibly satisfy this differential equation?*

A: Such functions **do** exist !

For example, the functions $V(z) = e^{+\gamma z}$ and $V(z) = e^{-\gamma z}$ each satisfy this transmission line wave equation (**insert** these into the differential equation and see for **yourself**!).

Likewise, since the transmission line wave equation is a **linear** differential equation, a weighted **superposition** of the two solutions is **also a solution** (again, **insert** this solution to and see for **yourself**!):

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

In fact, it turns out that **any and all** possible solutions to the differential equations can be expressed in **this** simple form!

Therefore, the **general** solution to these complex wave equations (and thus the telegrapher equations) are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** (V_0^+, V_0^-, I_0^+, I_0^-)!!

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z)$$

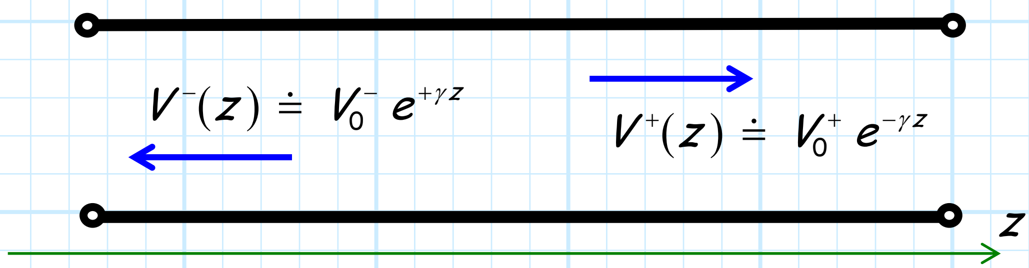
$$I(z) = I^+(z) + I^-(z)$$

where:

$$V^+(z) \doteq V_0^+ e^{-\gamma z} \qquad V^-(z) \doteq V_0^- e^{+\gamma z}$$

$$I^+(z) \doteq I_0^+ e^{-\gamma z} \qquad I^-(z) \doteq I_0^- e^{+\gamma z}$$

The two terms in each solution describe **two waves** propagating in the transmission line, **one** wave ($V^+(z)$ or $I^+(z)$) propagating in one direction ($+z$) and the **other** wave ($V^-(z)$ or $I^-(z)$) propagating in the **opposite** direction ($-z$).



Q: *So just what* ***are*** *the complex values* V_0^+, V_0^-, I_0^+, I_0^- ?

A: Consider the wave solutions at **one** specific point on the transmission line—the point $z = 0$. For example, we find that:

$$\begin{aligned} V^+(z=0) &= V_0^+ e^{-\gamma(z=0)} \\ &= V_0^+ e^{-(0)} \\ &= V_0^+ (1) \\ &= V_0^+ \end{aligned}$$

In other words, V_0^+ is simply the **complex** value of the wave function $V^+(z)$ **at the point z = 0** on the transmission line!

Likewise, we find:

$$V_0^- = V^-(z=0)$$

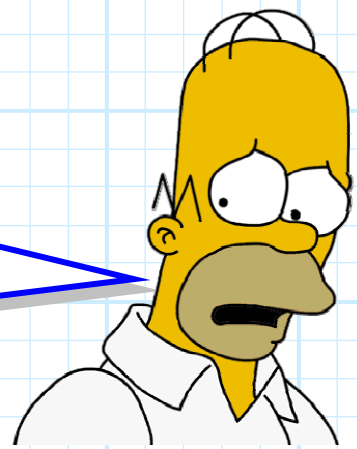
$$I_0^+ = I^+(z=0)$$

$$I_0^- = I^-(z=0)$$

Again, the four complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z)$, $I^+(z)$, $V^-(z)$, $I^-(z)$.

Q: *But what **determines** these wave functions? How do we **find** the values of constants V_0^+, I_0^+, V_0^-, I_0^-?*



A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active sources and/or passive loads)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!